MATH 141: Midterm 1

Name: Key

Directions:

- Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
- If you do not know how to solve a problem, try your best and/or explain in English what you would do.
- Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
		70

Legend

— my work

— or — my thoughts while I work

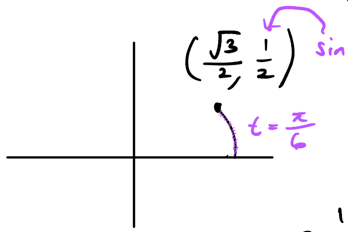
— Common mistakes made to avoid

1. If

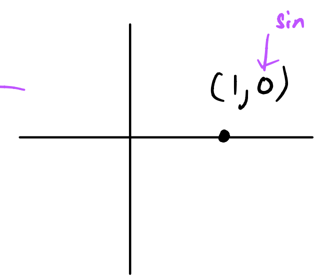
$$f(x) = x^2 - x \qquad g(x) = 3x^2 - x + 1 \qquad h(x) = \sin(x) \qquad j(x) = 2^x$$

Evaluate, expand, and/or simplify the following:

(a) $h\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \boxed{\frac{1}{2}}$



(b) $j(1) \cdot h(0) = 2^1 \cdot \sin(0)$



$= 2 \cdot 0$

$= \boxed{0}$

(c) $\underbrace{f(x)}_{\text{two term}} \cdot \underbrace{g(x)}_{\text{three term.}}$

Don't forget parenthesis when multiplying into ≥2 terms!

$$f(x) \cdot g(x) = (x^2 - x)\underbrace{(3x^2 - x + 1)}\ \underset{\text{law}}{\overset{\text{dist}}{=}}\ \underbrace{x^2}(3x^2 - x + 1) + \underbrace{(-x)}(3x^2 - x + 1)$$

$$\overset{\text{dist law}}{=} 3x^4 \boxed{-x^3} + \boxed{x^2} \boxed{-3x^3} + \boxed{x^2} - x$$

$$= \boxed{3x^4 - 4x^3 + 2x^2 - x}$$

(d) $f(x + h) - f(x)$

Since $f(x) = x^2 - x$

look! x+h replaces the "x" visually! Now do it!

$$f(\boxed{x+h}) - f(x) = \underbrace{(x+h)^2 - (x+h)}_{f(x+h)} - \underbrace{(x^2 - x)}_{f(x)}$$

Common mistake: forgot the parenthesis!

$$\underset{\text{dist law}}{\overset{\text{expand,}}{=}} x^2 + 2xh + h^2 - x - h - x^2 + x$$

$$= 2xh + h^2 - h$$

$$\overset{\text{GCF}}{=} \boxed{h(2x + h - 1)}$$

2. Short answer questions:

(a) Explain in English the intuition (not the definition) behind the symbols $\lim_{x \to a} f(x) = L$.

$f(x)$ is the height of the function at an x-value.

So as the x-values approach a but never a itself, the heights $f(x)$ will get closer and closer to the length of L.

(b) True or false: We can simplify

$$\frac{\underline{3(x-2)^2(x+3)} - 4(x+2)(x-3)^2}{5x(x-3)^2(x-2) - \underline{4(x+3)}}$$

by crossing out the $x + 3$.

No. $(x+3)$ is not a factor in the global context of both the numerator and denominator. The context in which it's a factor is underlined in purple above.

(c) If $f(x) = x - x^2$, evaluate $f(x + h)$ and fully expand + simplify.

this replaces x!

$$f\left(\boxed{x+h}\right) = (x+h) - \overset{(A+B)^2}{(x+h)^2}$$

Compare the notations:

$f(x+h)$

$f(x)$

look how $x+h$ took the place of x.

$$= x + h - (x^2 + 2xh + h^2) \overset{\text{dist}}{=} \boxed{x + h - x^2 - 2xh - h^2}$$

(d) If $F(x) = \sin^3(x^2)$ find three functions f, g, h where $f \circ g \circ h = F$.

$$\boxed{\begin{aligned} f(x) &= x^3 \\ g(x) &= \sin(x) \\ h(x) &= x^2 \end{aligned}}$$

Verifying:

deal w/ $h(x)$ first, we know what it is.

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

$$= f(\underline{g(x^2)})$$

x^2 takes the place of x in $g(x)$

$$= f(\overline{\sin(x^2)})$$

$\sin(x^2)$ replaces x in $f(x)$

$$= (\sin(x^2))^3 = \sin^3(x^2) = F(x)$$

Common mistake #1: plugging in 2 into $-x^2+1$ is $-2^2+1 = (-1)\cdot 2^2 + 1 = -4+1 = \boxed{-3}$ negative law #1.

3. Suppose

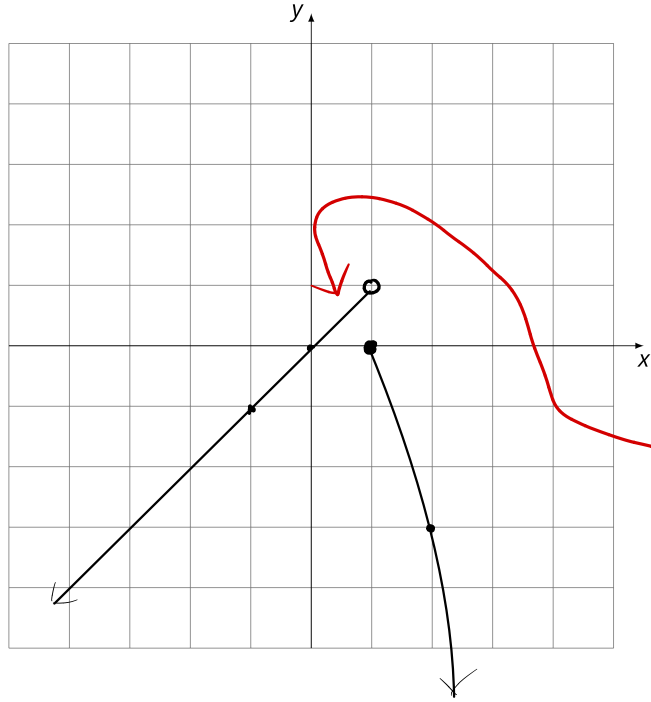
$$f(x) = \begin{cases} x & \boxed{x < 1} \\ \boxed{-x^2+1} & x \geq 1 \end{cases}$$

Common mistake #2: $x<1$ is

lots of people forgot to take this branch on the interval $(0,1)$. Which means $\boxed{\text{this}}$ part was forgotten.

(a) Sketch a graph of $f(x)$.

x	$f(x)$
-1	-1
0	0
1	$-1^2+1=0$
2	$-2^2+1=-3$
3	$-3^2+1=-8$



(b) What is $f(1)$?

$$f(1) = -1^2+1 \overset{\text{negative}}{\underset{\text{law 1}}{=}} (-1)\cdot 1^2 + 1 = -1+1 = \boxed{0}$$

(c) Does $\lim\limits_{x\to 1} f(x)$ exist? If it does, find the limit. If not, explain why.

No, because looking @ the graph above:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = \boxed{1}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x^2+1 = \boxed{0}$$

Since $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1} f(x)$ DNE.

4. Perform the given instruction. Remember to use the relevant laws/properties and **fully simplify**.

(a) Expand and simplify: $\dfrac{3(x+h)^2-1-(3x^2-1)}{h}$

common mistake:

$3(x+h)^2 \neq (3x+3h)^2$

because $3(x+h)^2 = 3\cdot(x+h)\cdot(x+h)$

You can only distribute the 3 to one factor of $(x+h)$.

3 multiplies into 3 terms.

Don't forget parenthesis.

$$\frac{3\overbrace{(x+h)^2}^{(A+B)^2}-1-(3x^2-1)}{h} \overset{\text{dist}}{=} \frac{3(x^2+2xh+h^2)\cancel{-1}-3x^2\cancel{+1}}{h}$$

$$\overset{\text{dist}}{=} \frac{\cancel{3x^2}+6xh+3h^2\cancel{-3x^2}}{h}$$

$$= \frac{6xh+3h^2}{h} \overset{\text{GCF}}{=} \frac{\cancel{h}(6x+3h)}{\cancel{h}} \underset{\text{law 5}}{\overset{\text{frac}}{=}} \boxed{6x+3h}$$

(b) Rationalize the numerator (remember to simplify): $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

the only technique to square two terms.

$$(A-B)\cdot(A+B) = A^2-B^2$$

$$\frac{\overbrace{\sqrt{x+h}}^{A}-\overbrace{\sqrt{x}}^{B}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$\underset{\text{law 5}}{\overset{\text{frac}}{=}} \boxed{\frac{1}{\sqrt{x+h}+\sqrt{x}}}$$

mistake #1

$$\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)^2 \neq \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h^2}$$

because the numerator are terms and exponents don't interact with terms.

mistake #2

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot \sqrt{x+h}+\sqrt{x}$$

forgot denominator $\sqrt{x+h}+\sqrt{x}$

Compound fraction.
Technique: get rid of nested denominators by multiplying by LCD of nested denominators.

LCD of $\frac{2}{x(x+1)}, \frac{3}{\sqrt{x}}, \frac{1}{x}$ is $\boxed{x\sqrt{x}(x+1)}$

We begin: (c) Simplify: $\dfrac{\boxed{\dfrac{2}{x^2+x}} - \boxed{\dfrac{3}{\sqrt{x}}}}{\sqrt{x}+\boxed{\dfrac{1}{x}}}$ $\quad \frac{2}{x(x+1)}$

$$\frac{\overbrace{\frac{2}{x^2+x}-\frac{3}{\sqrt{x}}}^{\text{two terms.}}}{\sqrt{x}+\frac{1}{x}} \cdot \frac{x\sqrt{x}(x+1)}{x\sqrt{x}(x+1)} \underset{\text{law 1.}}{\overset{\text{frac}}{=}} \frac{\left(\frac{2}{x(x+1)}-\frac{3}{\sqrt{x}}\right)x\sqrt{x}(x+1)}{\left(\sqrt{x}+\frac{1}{x}\right)x\sqrt{x}(x+1)}$$

two terms

two terms

$$\underset{\text{law}}{\overset{\text{distributive}}{=}} \frac{\frac{2}{\cancel{x(x+1)}}\cdot \cancel{x}\sqrt{x}\cancel{(x+1)} - \frac{3}{\cancel{\sqrt{x}}}x\cancel{\sqrt{x}}(x+1)}{x(\sqrt{x})^2(x+1) + \frac{1}{\cancel{x}}\cancel{x}\sqrt{x}(x+1)}$$

$$\underset{\text{1, then 5}}{\overset{\text{fraction law}}{=}} \frac{2\sqrt{x}-3x(x+1)}{x^2(x+1)+\sqrt{x}(x+1)}$$

$$x^{\frac{1}{2}}\cdot(x+1) = x^{\frac{1}{2}}\cdot x + x^{\frac{1}{2}} = x^{\frac{3}{2}} + x^{\frac{1}{2}} = \sqrt{x^3}+\sqrt{x}$$

$$= \boxed{\frac{2\sqrt{x}-3x^2-3x}{x^3+x^2+\sqrt{x^3}+\sqrt{x}}}$$

you could have also factored out $x^{\frac{1}{2}}$ from numerator and denominator, then cancelled.

(d) $\boxed{\text{Expand}}$: $(x^3+6)(2x+1)-(x^2+x-2)(3x^2)$

Convert to terms, no parenthesis.

$$(x^3+6)(2x+1)-(x^2+x-2)(3x^2) \overset{\text{dist}}{=} (x^3+6)2x+(x^3+6)\cdot 1-3x^4-3x^3+6x^2$$

$$\overset{\text{dist}}{=} 2x^4+12x+x^3+6-3x^4-3x^3+6x^2$$

$$= \boxed{-x^4-2x^2+6x^2+12x+6}$$

Common mistake: forgetting to distribute the factor of (-1)

you must pass the Vertical Line Test.

5. Draw the graph of a function which satisfies the following:

(a) $f(0) = 1$

(b) $f(2) = 1$

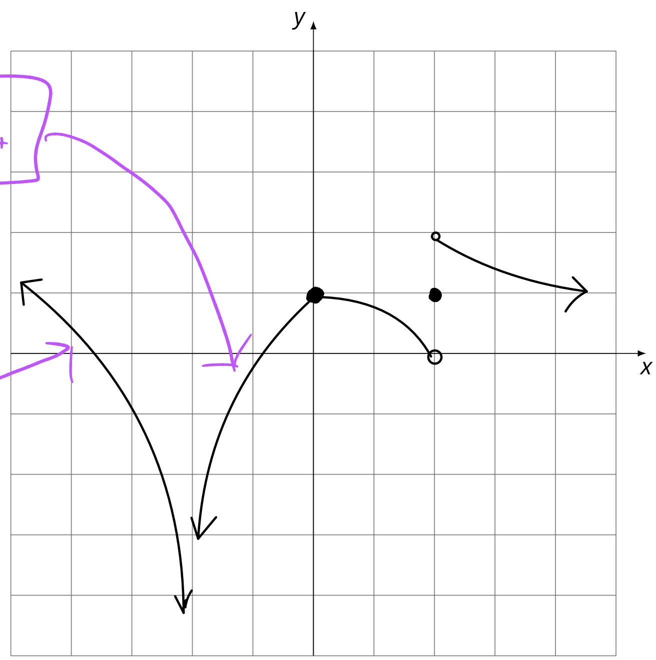
(c) $\lim_{x \to 0} f(x) = 1$

(d) $\lim_{x \to 2^-} f(x) = 0$

(e) $\lim_{x \to 2^+} f(x) = 2$

(f) $\boxed{\lim_{x \to -2} f(x) = -\infty}$

both $\boxed{\lim_{x \to -2^-}}$ and $\boxed{\lim_{x \to -2^+}}$ must agree.



6. Consider this limit:

$$\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

(a) Try using Limit Laws to find the limit. What ends up happening?

$$\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} \underset{①,②}{\overset{\text{limit law } ⑤}{=}} \frac{\frac{\lim_{h\to 0} 1}{\lim_{h\to 0} 3 + \lim_{h\to 0} h} - \lim_{h\to 0}\left(\frac{1}{3}\right)}{\lim_{h\to 0} h}$$

$$\overset{\text{limit law } ⑦,⑧}{=} \frac{\frac{1}{3+0} - \frac{1}{3}}{0}$$

$$= \frac{0}{0}$$

You end up with an indeterminate form of type $\frac{0}{0}$.

(b) Now find the actual limit.

The $\lim_{h\to 0}$ says you're looking to create a global factor of $h - 0 = \boxed{h}$ in the numerator. So, simplify the compound fraction.

$$\lim_{h\to 0} \frac{\frac{1}{(3+h)} - \frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3}{3}\frac{1}{(3+h)} - \frac{1}{3}\cdot\frac{3+h}{3+h}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{3(3+h)} - \frac{3+h}{3(3+h)}}{h} \quad \text{frac law ①}$$

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h} \quad \text{frac law ③}$$

$$= \lim_{h\to 0} \frac{\frac{3-3-h}{3(3+h)}}{h} \quad \text{dist law}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{3(3+h)} \cdot \frac{1}{h} \quad \text{frac law ②}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{3\cdot\cancel{h}\cdot(3+h)} \quad \text{frac law ⑤}$$

$$= \lim_{h\to 0} \frac{-1}{3(3+h)} \quad \text{now use limit laws}$$

$$= \frac{-1}{3(3+0)}$$

$$= \boxed{-\frac{1}{9}}$$

7. Use the **mathematical definition of continuity** to prove the function

$$f(x) = \begin{cases} x(x-1) & x < 1 \\ 0 & x = 1 \\ \sqrt{x-1} & x > 1 \end{cases}$$

is continuous at the number $x = 1$.

① Show $\lim_{x \to 1} f(x)$ exists.

limit laws

we have: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x-1} = \sqrt{\lim_{x \to 1^+} x - \lim_{x \to 1^+} 1} = \sqrt{1-1} = 0$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} [x(x-1)] = \left[\lim_{x \to 1^-} x\right]\left[\lim_{x \to 1^-} x - \lim_{x \to 1^-} 1\right]$

limit laws

$= 1 \cdot (1-1)$

$= 0$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, we conclude $\lim_{x \to 1} f(x)$ exists and is equal to 0.

② Show $f(1)$ is defined.

$f(1) = 0$ ✓

③ Show $\lim_{x \to 1} f(x) = f(1)$

from parts ① and ②, $\lim_{x \to 1} f(x) = 0$ and $f(1) = 0$.

∴, this condition is satisfied.

By the definition of continuity $f(x)$ is continuous at $x = 1$.